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Stability of global strong solutions of the Navier-Stokes equations

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0. Introduction

We consider the following Navier-Stokes system.

$$(NS) \quad \begin{cases} u_t - \Delta u + (u \cdot \nabla)u + \nabla \pi = 0 & \text{in } \mathbf{R}^N \times \mathbf{R}^+, \\ \nabla \cdot u = 0 & \text{in } \mathbf{R}^N \times \mathbf{R}^+, \\ u(x, 0) = u_0(x) & \text{in } \mathbf{R}^N. \end{cases}$$

Let P be the Helmholtz projection. We denote by $\|\cdot\|_p$ the norm of $L^p(\mathbf{R}^N)$. Kato [K] showed that for any $u_0 \in PL^N$ the problem (NS) has a unique local (strong) solution $u(t; u_0) \in C([0, T]; PL^N) \cap L^{N+2}((0, T); PL^{N+2})$ (, where $T = T(\|u_0\|_N) > 0$) and that $T = \infty$ and $u(t; u_0) \in C_0([0, \infty); PL^N) := \{u \in C([0, \infty); PL^N); \lim_{t \rightarrow \infty} \|u(t)\|_N = 0\}$ if $\|u_0\|_N$ is sufficiently small.

We study the stability of global solutions of (NS) belonging to $C_0([0, \infty); PL^N)$. This class of solutions are very important since all strong global solution belongs to $C_0([0, \infty); PL^N)$ provided $2 \leq N \leq 4$ and $u_0 \in PL^2 \cap PL^N$ (see Section 3).

1. Navier-Stokes system

First we will characterize the global solutions belonging to $C_0([0, \infty); PL^N)$.

Proposition 1.1. *Let u be a global solution of (NS) with the initial value $u_0 \in PL^N$. Then we have the following.*

(i) *If $u \in C_0([0, \infty); PL^N)$ then we have $u \in L^q(\mathbf{R}^+; PL^r)$ with $1/q = 1/2 - N/2r$, where $q > N$ and $r > N$.*

(ii) *Let r be a constant such that $N < r \leq 2N$. If $u \in L^q(\mathbf{R}^+; PL^r)$ with $1/q = 1/2 - N/2r$ then we have $u \in C_0([0, \infty); PL^N)$.*

Proof. We can easily derive (i) from [K, Theorems 1 and 2] by Kato. We omit the proof of (ii) since it was implicitly given in the proof of [K, Theorem 2']. ■

Remark 1.1. When $N = 3$, Ponce et al [PRST] obtained a similar result under an assumption: $u_0 \in PL^2 \cap H^1$.

Theorem 1.1. Let $u(t; u_0) \in C_0([0, \infty); PL^N)$ be a global solution of (NS). Then there exists a constant $\delta \in \mathbf{R}^+$ depending only on N and u_0 such that if

$$(1.1) \quad v_0 \in PL^N \quad \text{and} \quad \|v_0 - u_0\|_N \leq \delta$$

then (NS) has a unique global solution $u(t; v_0)$ satisfying

$$(1.2) \quad \|u(t; v_0) - u(t; u_0)\|_N \leq \|v_0 - u_0\|_N \exp\left(C_1 \int_0^t \|u(s; u_0)\|_{N+2}^{N+2} ds\right) \quad \text{for } t \geq 0,$$

where the constant $C_1 \in \mathbf{R}^+$ depends only on N .

We have an immediate corollary of Theorem 1.1:

Corollary 1.1. For (NS) we set

$$A = \{u_0 \in PL^N; u(t; u_0) \in C_0([0, \infty); PL^N)\}.$$

Then A is open in PL^N .

Remark 1.2. When $N = 3$, the set A is unbounded in PL^3 (see [UI]).

Our Theorem 1.1 extends [Wi, Theorem 1] and [PRST, Theorem 1]. Wiegner [Wi] obtained a $L^2 \cap L^r$ -stability result with $r > N$. Ponce et al [PRST] obtained a H^1 -stability result for $N = 3$.

Proof of Theorem 1.1. Our proof is close to the argument in [N] and [Ka1] for the porous media equations. We denote $\partial_j := \partial/\partial x_j$. We have a (unique) local strong solution $u(t; v_0) \in C([0, T]; PL^N)$. We will derive the estimate (1.2). Set $w(t) := u(t; v_0) - u(t; u_0)$ and $u := u(t; u_0)$ for simplicity. Then w satisfies

$$(1.3) \quad \begin{cases} w_t - \Delta w + (w \cdot \nabla)w + (u \cdot \nabla)w + (w \cdot \nabla)u + \nabla \pi = 0 & \text{in } \mathbf{R}^N \times \mathbf{R}^+, \\ \nabla \cdot w = 0 & \text{in } \mathbf{R}^N \times \mathbf{R}^+. \end{cases}$$

By integration by parts,

$$(1.4) \quad \frac{1}{p} \frac{d}{dt} \int_{\mathbf{R}^N} |w(t)|^p = -A_p(w)^2 - (p-2)B_p(w)^2 - I_1 - I_2 - I_3 - I_4,$$

where we set

$$\begin{aligned} A_p(w) &= \left(\int |\nabla w|^2 |w|^{p-2} \right)^{1/2}, \\ B_p(w) &= \left(\int |\nabla |w||^2 |w|^{p-2} \right)^{1/2}, \\ I_1 &= \int |w|^{p-2} w \cdot (w \cdot \nabla) w, \\ I_2 &= \int |w|^{p-2} w \cdot (w \cdot \nabla) u, \\ I_3 &= \int |w|^{p-2} w \cdot (u \cdot \nabla) w, \\ I_4 &= \int |w|^{p-2} w \cdot \nabla \pi. \end{aligned}$$

With the aid of Gagliardo - Nirenberg inequality,

$$(1.5) \quad \|w\|_{p+2} \leq C_p \|w\|_N^{2/(p+2)} A_p(w)^{2/(p+2)}.$$

In what follows, we always set $p = N$. We will estimate I_j with $j = 1, 2, 3, 4$. By (1.5),

$$(1.6) \quad \begin{aligned} |I_1| &\leq \int |w|^N |\nabla w| \leq C_\varepsilon \int |w|^{N+2} + \varepsilon A_p(w)^2 \\ &\leq (C \|w\|_N^2 + \varepsilon) A_N(w)^2. \end{aligned}$$

It follows from the integration by parts and (1.5) that

$$(1.7) \quad \begin{aligned} |I_2| + |I_3| &\leq N \int |u| |w|^{N-1} |\nabla w| \\ &\leq \varepsilon \int |w|^{N-2} |\nabla w|^2 + C_\varepsilon \int |u|^2 |w|^N \\ &\leq \varepsilon A_N(w)^2 + C \|u\|_{N+2}^2 \|w\|_{N+2}^N \\ &\leq \varepsilon A_N(w)^2 + C \|u\|_{N+2}^2 \|w\|_N^{2N/(N+2)} A_N(w)^{2N/(N+2)} \\ &\leq 2\varepsilon A_N(w)^2 + C \|u\|_{N+2}^{N+2} \|w\|_N^N. \end{aligned}$$

By similar argument in [VS] we will estimate I_4 . In view of (1.3) we have

$$(1.8) \quad -\Delta \pi = \sum_{i,j} \partial_j w^i \cdot \partial_i (2u^j + w^j) = \sum_{i,j} \partial_i \partial_j [w^i (2u^j + w^j)].$$

By the Calderon - Zygmund inequality and Hölder's inequality,

$$(1.9) \quad \|\pi\|_{(N+2)/2}^2 \leq C \sum_{i,j} \|w^i(2u^j + w^j)\|_{(N+2)/2}^2 \leq C \|w\|_{N+2}^2 (\|u\|_{N+2}^2 + \|w\|_{N+2}^2).$$

It follows from the integration by parts, (1.5), (1.7) and (1.9) that

$$(1.10) \quad \begin{aligned} |I_4| &\leq (N-2) \int |\pi| |w|^{N-2} |\nabla w| \\ &\leq C_\varepsilon \int |\pi|^2 |w|^{N-2} + \varepsilon \int |w|^{N-2} |\nabla w|^2 \\ &\leq C \|\pi\|_{(N+2)/2}^2 \|w\|_{N+2}^{N-2} + \varepsilon A_N(w)^2 \\ &\leq C \|w\|_{N+2}^N (\|u\|_{N+2}^2 + \|w\|_{N+2}^2) + \varepsilon A_N(w)^2 \\ &\leq C_\varepsilon \|u\|_{N+2}^{N+2} \|w\|_N^N + (2\varepsilon + C \|w\|_N^2) A_N(w)^2. \end{aligned}$$

Therefore, we have

$$(1.11) \quad \frac{1}{N} \frac{d}{dt} \|w(t)\|_N^N \leq -\left(\frac{1}{2} - C_0 \|w\|_N^2\right) A_N(w)^2 + C_1 \|u\|_{N+2}^{N+2} \|w\|_N^N.$$

Set $\delta := (2C_0)^{-1/2} \exp(-C_1 \int_0^\infty \|u(s; u_0)\|_{N+2}^{N+2})$. Let $\|v_0 - u_0\|_N \leq \delta$. Then we obtain

(1.2) from (1.11). ■

Remark 1.2. Although we can obtain some similar results for the Dirichlet problem for the Navier-Stokes system, we omit here. See [K3] for the details.

2. Scalar semilinear heat equation

We can obtain some results of a semilinear heat equation in the similar argument in Section 1. We consider the following problem (H).

$$(H) \quad \begin{cases} u_t = \Delta u + |u|^{p-1} u & \text{in } \mathbf{R}^N \times \mathbf{R}^+, \\ u(x, 0) = u_0(x) & \text{in } \mathbf{R}^N, \end{cases}$$

where $p \in (1 + 2/N, \infty)$. We set $p_0 := N(p-1)/2 (> 1)$. Giga [G] showed that for any $u_0 \in L^{p_0}$ the problem (H) has a unique local solution $u(t; u_0) \in C([0, T); L^{p_0}) \cap L^{p_0+p-1}((0, T); L^{p_0+p-1})$ (, where $T = T(\|u_0\|_{p_0}) > 0$) and that $T = \infty$ and $u(t; u_0) \in C_0([0, \infty); L^{p_0})$ if $\|u_0\|_{p_0}$ is sufficiently small.

We state our results without proofs. See [Ka3] for the proofs.

Proposition 2.1. *Let u be a global solution of (H) with the initial value $u_0 \in L^{p_0}$.*

Then we have the following.

(i) *If $u \in C_0([0, \infty); PL^N)$ then we have $u \in L^q(\mathbf{R}^+; PL^r)$ with $1/q = (1/p_0 - 1/r)N/2$, where $q > \max(p_0, p)$ and $r > p_0$.*

(ii) *Let r be a constant such that $r > p_0$ and $p \leq r \leq p_0 p$. If $u \in L^q(\mathbf{R}^+; PL^r)$ with $1/q = (1/p_0 - 1/r)N/2$ then we have $u \in C_0([0, \infty); L^{p_0})$.*

Theorem 2.1. *Let $u(t; u_0) \in C_0([0, \infty); L^{p_0})$ be a global solution of (H). Then there exists a constant $\delta \in \mathbf{R}^+$ depending only on N, p and u_0 such that if*

$$v_0 \in L^{p_0} \quad \text{and} \quad \|v_0 - u_0\|_{p_0} \leq \delta$$

then (H) has a unique global solution $u(t; v_0)$ satisfying

$$\|u(t; v_0) - u(t; u_0)\|_{p_0} \leq \|v_0 - u_0\|_{p_0} \exp\left(C_1 \int_0^t \|u(s; u_0)\|_{\frac{p_0+p}{p_0+p-1}}^{p_0+p-1} ds\right) \quad \text{for } t \geq 0,$$

where the constant $C_1 \in \mathbf{R}^+$ depends only on N .

We have an immediate corollary of Theorem 2.1:

Corollary 2.1. *For (H) we set*

$$A = \{u_0 \in L^{p_0}; u(t; u_0) \in C_0([0, \infty); L^{p_0})\}.$$

Then A is open in L^{p_0} .

By [Ka2] and [Wa] we see that the set A is unbounded in L^{p_0} .

Remark 2.1. Our Theorem 2.1 extends [Ka2, Proposition 6] by the author, where we assumed $u_0 \in L^1 \cap L^\infty$ and $u_0 \geq 0$ in \mathbf{R}^N .

3. Structure of space of solutions for (NS) and (H)

We mention the topological structure for the space of solutions of (NS) and (H).

We set

$$B := \{u_0 \in PL^N; \|u(t; u_0)\|_N \text{ blows up in finite time} \} \quad \text{for (NS)}$$

and

$$B := \{u_0 \in L^{p_0}; \|u(t; u_0)\|_{p_0} \text{ blows up in finite time} \} \quad \text{for (H)}.$$

For (NS) we have $A = PL^2$ for $N = 2$ (see [KM], [M] and [Wi]). However, we can easily derive this well-known result from our Proposition 1.1 and the energy equality. Indeed, by the energy equality:

$$(3.1) \quad \|u(t)\|_2^2 + 2 \int_0^t \|\nabla u(s)\|_2^2 ds = \|u_0\|_2^2,$$

$u(t; u_0)$ is global for any $u_0 \in PL^2$. By Gagliardo-Nirenberg inequality

$$(3.2) \quad \|u(t)\|_4 \leq C \|u(t)\|_2^{1/2} \|\nabla u(t)\|_2^{1/2}.$$

In view of (3.1) and (3.2) we have $u(t; u_0) \in L^4(\mathbf{R}^+; PL^4)$. Therefore, we immediately obtain $u(t; u_0) \in C_0([0, \infty); L^2)$ from our Proposition 1.1 (ii). For (NS) we have $(A \cup B) \cap PL^2 = PL^2 \cap PL^N$ for $N = 3$ and $N = 4$, which is due to the energy equality. Therefore, $B \cap PL^2$ is closed in $PL^2 \cap PL^N$. Since $PL^2 \cap PL^N$ is dense in PL^N , we see that for $N = 3$ and $N = 4$ the set B is empty or B is not open in PL^N .

It seems to be interesting to compare (NS) with (H). If u is a solution of (H) with $p = 3$ then $\lambda u(\lambda x, \lambda^2 t)$ is also a solution of (H) for $\lambda > 0$. We remark that the solution of (NS) has just the same property with respect to the same self-similar transformation. For (H) with $p = 3$ we have the following (see [Ka2]): the set $B \cap L^2$ is not empty and is open in $L^2 \cap L^N$. If we set $S := \{u_0 \in L^2 \cap L^N - (A \cup B); u_0(x) \geq 0 \text{ in } \mathbf{R}^N\}$ then S is not empty and $S \subset \partial A$, where ∂A is the boundary of A in L^N .

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